Self-consistent equations for critical exponents of the Grassmannian sigma -model

1991 J. Phys. A: Math. Gen. 24 L243

(http://iopscience.iop.org/0305-4470/24/5/006)

J. Phys. A: Math. Gen. 24 (1991) L243–L248. Printed in the UK

LETTER TO THE EDITOR

Self-consistent equations for critical exponents of the Grassmannian σ-model

Yu M Pis'mak†§ and M V Polyakov‡

† Research Institute for Theoretical Physics, University of Helsinki, Siltavuorenpenger 20 C, SF-00170 Helsinki, Finland

‡ Leningrad Nuclear Physics Institute, Gatchina, USSR

Received 1 January 1991

Abstract. A method of calculating the critical exponents based on skeleton self-consistent equations is used for calculation of the exponents ν and η of a Grassmannian nonlinear σ-model of symmetry $G(2K)/G(K)\times G(K)$ with $K=0$. These exponents are related to the conductivity and the participation ratio exponents at the mobility edge in the Anderson localization problem.

Some time ago a nonlinear σ-model was proposed for the description of the Anderson transition [1]. In this model fields have values on Grassmannian manifolds $G(2K)/G(K)\times G(K)$, where G is the unitary, orthogonal or symplectic group and K is a number of replicas, which tends to zero eventually. The quantities (conductance, (inverse) participation ratio and so on), which distinguish between the regime of localized and extended states are connected with relevant scaling operators of the σ-model. Calculations of their critical exponents in the framework of an ε-expansion [2, 3] show that the higher-order corrections are important. For instance, the ε-expansion of the exponent ν yields $\nu=0.37$ (at $d=3$), which is too small, since it violates the inequality $\nu\geqslant 2/d$ proved for disordered systems [4]. Most probably, the usual technique of the ε-expansion must be significantly modified for Grassmannian σ-models [3, 5]. Therefore it would be useful to study these σ-models by other methods.

In the present letter we calculate the critical exponents of the Grassmannian σ-models by the method of self-consistent equations [6]. This method has been used for calculations of the critical exponents [7–9] of the scalar nonlinear σ-model.

Exactly at the critical point all Green functions are scale invariant, so a propagator has a simple power-law behaviour. For instance, in coordinate space a scalar propagator is of the form $G(x)=A/|x|^{2\alpha}$. The unknown exponent α is determined by skeleton equations for propagators and vertices without bare terms [6–9]. These equations contain infinite sums of skeleton diagrams. Sometimes their solutions can be found as a series in some parameter. In the case of the $O(N)$ σ-model such a parameter is $1/N$. The critical exponent η of the $O(N)$ σ-model has been calculated up to order $1/N^3$ in such a way [9]. For the Grassmannian σ-model with zero number of replicas there is no such parameter. However, we may exploit the fact that terms of self-consistent equations are functions of the critical exponents. In particular they depend on vertex

§ Permanent address: Department of Theoretical Physics, State University of Leningrad, Staryi Petergof, Leningrad, USSR.

0305-4470/91/050243+06$03.50

anomalous dimensions (e.g. $\kappa=(1/\nu)+2-d-\eta$), which are assumed to be small. Therefore we replace each term of the equations by its leading order approximation in these parameters. For models of the Anderson transition recent numerical calculations of the exponent ν [10] yield values close to 1 ($\nu=0.9\pm0.3$ for the Gauss distribution, and $\nu=1.4\pm0.2$ for the box distribution). So one may hope that $\kappa=(1/\nu)+2-d-\eta$ is indeed numerically small at $d=3$.

The scalar nonlinear σ-model is described by the Lagrangian

$$L=-\frac{1}{2t}\partial_\mu\phi^a\partial_\mu\phi^a$$

where ϕ^a are boson fields ($a=1,\ldots,N$) with the constraint $\phi^a(x)\phi^a(x)=N$. Introducing an auxiliary field ψ one obtains the equivalent model with the Lagrangian

$$L'=-\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}\psi(\phi^2-tN)$$

where there are no constraints on the fields.

As mentioned above, at the critical point there is a system of self-consistent equations for the dressed propagators D_ϕ, D_ψ and the vertex $V_{\psi\phi\phi}$. These equations are depicted in figure 1, where the straight lines denote the propagator D_ϕ, and the wavy line D_ψ. In the coordinate space the arrow pointing from x to y corresponds to the vector $(y-x)$. The shaded boxes in the Dyson equations for propagators denote the kernels of the Bethe–Salpeter equation.

According to the scaling hypothesis all Green functions in the critical region are scale invariant. The scale invariance determines the propagators up to constant amplitudes, which can be chosen equal to 1 by the normalization conditions of the fields

$$D_\phi^{ab}(x)=\frac{\delta^{ab}}{|x|^{2\alpha}}\qquad D_\psi(x)=\frac{1}{|x|^{2\beta}}.$$

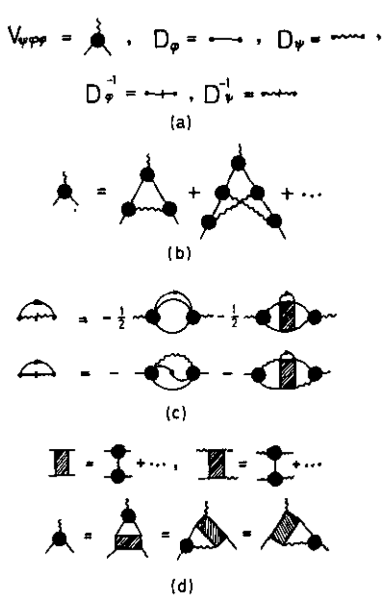

Figure 1. Self-consistent equations: (*a*) notation; (*b*) vertex equation; (*c*) Dyson equations; (*d*) Bethe–Salpeter kernels and their relations with the vertex.

The inverse propagators also have a power-law behaviour:

$$D_\phi^{-1}(x)=\frac{p(\alpha)}{|x|^{2(2\mu-\alpha)}} \qquad D_\psi^{-1}(x)=\frac{p(\beta)}{|x|^{2(2\mu-\beta)}}.$$

Here we use the notation

$$p(x)=\pi^{-2\mu}a(\mu-x)a(x-\mu) \qquad a(x)=\frac{\Gamma(\mu-x)}{\Gamma(x)}$$

and $\mu=d/2$. The exponents α and β are related to the conventional ones ν and η via

$$\alpha=\mu-1+\frac{\eta}{2} \qquad \beta=2-\eta-\kappa \qquad \frac{1}{\nu}=2\mu-\beta.$$

The dressed vertex $V_{\psi\phi\phi}(y,x,x')$ is scale invariant; hence

$$\int \mathrm{d}y\ V_{\psi\phi\phi}(y,x,x')=\frac{\Gamma(\mu)Z}{2\pi^{2\mu}|x-x'|^{2(\mu+\kappa/2)}} \tag{1}$$

where Z is a constant. If there is the conformal invariance at critical point, then the three-point vertex is determined up to a constant. This fact was used in [9], but here we do not use it since the conformal invariance is not evident for the Grassmannian σ-model at the critical point. Integrating the vertex equation over one coordinate we obtain the skeleton equation given in figure 2, where the cross denotes the integration over the coordinates of the corresponding point. It is easy to see that the graphs on the RHS of this equation diverge when κ tends to zero: these graphs have poles in κ. The first three-vertex graph can be easily calculated in the leading pole approximation ($\kappa\to 0$) using the technique of coordinate integrations [8, 9]. In this approximation the equation for the dressed vertex takes the form

$$Z=-\frac{2\pi^{2\mu}Z^3}{\Gamma(\mu)\kappa^3}a(\alpha)^2a(2-\eta)(1+\mathrm{O}(\kappa)).$$

We see that $Z=\mathrm{O}(\kappa^{3/2})$ and we can neglect graphs with more than three vertices in the leading order of the κ-expansion. This holds for the Dyson equations, too. Performing the κ-expansion on the RHS of these equations for propagators we obtain in the leading order

$$p(\alpha)=-\frac{Z^2}{\kappa^2}(1+\mathrm{O}(\kappa)) \qquad p(\beta)=-\frac{nZ^2}{2\kappa^2}(1+\mathrm{O}(\kappa)). \tag{2}$$

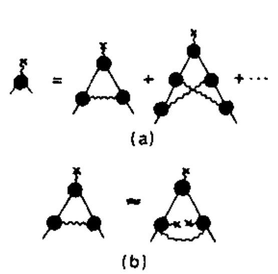

Figure 2. Auxiliary vertex equation: (a) the vertex equation integrated over the coordinates of the ψ-point; (b) the three-vertex term in the leading approximation.

We have three transcendental equations for three unknown variables η, κ and Z. Expanding the function $p(\beta)$ in κ we obtain from (2):

$$Z^2 = Q(\eta)\kappa^3$$

$$\kappa = -\frac{2p(2-\eta)}{nQ(\eta)-2p(2-\eta)(B(2-\eta)-B(2\mu-2+\eta))} \tag{3}$$

$$p\left(\mu-1+\frac{\eta}{2}\right) = Q(\eta)\frac{2p(2-\eta)}{nQ(\eta)-2p(2-\eta)(B(2-\eta)-B(2\mu-2+\eta))}$$

where

$$Q(\eta) = -\frac{\Gamma(\mu)}{2\pi^{2\mu}a(\mu-1+\eta/2)^2a(2-\eta)}$$

$$B(x) = \psi(x)+\psi(\mu-x) \qquad \psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x).$$

For the case of the 3D Ising model ($N=1$, $\mu=\frac{3}{2}$) we obtain from (3) $\eta=6.1\times10^{-2}$, $\kappa=0.2$, $\nu=0.79$, $Z^2=7.4\times10^{-5}$. For the 3D Heisenberg model ($N=3$, $\mu=\frac{3}{2}$) we have $\eta=4.3\times10^{-2}$, $\kappa=0.14$, $\nu=0.84$, $Z^2=7.4\times10^{-5}$. In the framework of the high-temperature expansion, calculations [11] yield for these quantities the following values: $\eta=0.055\pm0.010$, $\nu=0.638\pm0.002$ in the case of the 3D Ising model, and $\eta=0.04\pm0.003$, $\nu=0.71\pm0.02$ for the 3D Heisenberg model. It can be seen that our values of the exponent η are in a good agreement with the numerical results of [11]. As to the exponent ν our results are about 20% more than the corresponding values of ν in [11]. Our estimations show that the self-consistent equations in the next-to-leading order of the κ-expansion have solutions for exponents which are in better agreement with the exponents calculated in [11].

The Grassmannian σ-model with the unitary (orthogonal) symmetry is defined by the Lagrangian:

$$L = -\frac{1}{2t}\operatorname{tr}(\partial_\mu Q\partial_\mu Q)$$

where $Q(x)$ is a Hermitian (symmetric) $K\times K$ matrix with the constraints $\operatorname{tr} Q(x)=0$ and $Q(x)^2=1$. We are interested in critical exponents of this model with $K=0$. Let us parametrize the matrix $Q(x)$ in the following way:

$$Q(x) = \sum_{a=1}^{n(K)} \phi^a t^a$$

where t^a is a complete set of the traceless Hermitian (symmetric) matrices such that $\operatorname{tr}(t^at^b)=\delta^{ab}$, $n(K)=K^2-1((K^2+K-2)/2)$. In this parametrization the constraints on $Q(x)$ take the form:

$$\phi^a\phi^a = K \qquad d_{abc}\phi^b\phi^c = 0.$$

Here $d_{abc}=\operatorname{tr}(t^a[t^b,t^c]_+)$. Introducing auxiliary fields ψ^a and ρ we obtain the Lagrangian of the equivalent model

$$L' = -\tfrac{1}{2}(\partial\phi)^2+\tfrac{1}{2}d_{abc}\psi^a\phi^b\phi^c+\tfrac{1}{2}\rho(\phi^2-tK)$$

without any constraints on the fields. At the critical point its dressed propagators can be chosen in the form:

$$D_\phi^{ab}(x)=\frac{\delta^{ab}}{|x|^{2\alpha}} \qquad D_\psi^{ab}(x)=\frac{\delta^{ab}}{|x|^{2\gamma}} \qquad D_\rho(x)=\frac{1}{|x|^{2\beta}}.$$

The exponents α, γ and β are related to η, ν (exponent of the correlation length) and s (exponent of the conductivity) via

$$\alpha=\mu-1+\frac{\eta}{2} \qquad \beta=2-\eta-\kappa \qquad \gamma=2-\eta-\chi$$

$$\frac{1}{\nu}=2\mu-\beta \qquad s=(2\mu-2)\nu.$$

Here κ and χ are the anomalous dimensions of the dressed vertices $V_{\rho\phi\phi}$, $V_{\psi\phi\phi}$ respectively. As in the case of the $O(N)$ σ-model we solved the skeleton self-consistent equations for the dressed propagators D_ϕ, D_ρ, D_ψ and vertices $V_{\rho\phi\phi}$, $V_{\psi\phi\phi}$ in the leading approximation of the (κ, χ)-expansion. Using the relations

$$d_{aij}d_{bij}=f_1\delta_{ab} \qquad d_{aij}d_{bik}d_{cjk}=f_2 d_{abc}$$

where $f_1=2(K-4/K)$, $f_2=(K-12/K)$ for the Hermitian case, and $f_1=2(1+K/2-4/K)$, $f_2=(2+K/2-12/K)$ for the symmetric case, we obtain five equations for the exponents κ, η, χ and for two amplitudes Z_1, Z_2 similar to Z in (1). These equations are of the form:

$$1=\frac{1}{Q(\eta)\kappa}\left[\frac{Z_1^2}{\kappa^2}+\frac{Z_2^2}{\chi^2}f_1\right] \qquad 1=\frac{1}{Q(\eta)\chi}\left[\frac{Z_1^2}{\kappa^2}+\frac{Z_2^2}{\chi^2}f_2\right]$$

$$p\left(\mu-1+\frac{\eta}{2}\right)=-\left[\frac{Z_1^2}{\kappa^2}+\frac{Z_2^2}{\chi^2}f_2\right]$$

$$p(2-\eta)[1+\kappa(B(2\mu-2+\eta)-B(2-\eta))]=-\frac{n(K)Z_1^2}{2\kappa^2}$$

$$p(2-\eta)[1+\chi(B(2\mu-2+\eta)-B(2-\eta))]=-\frac{Z_2^2 f_1}{2\chi^2}.$$

In this approximation at $K=0$ we have a degeneracy of the exponents for models with the unitary and orthogonal symmetries. The degeneracy is lifted in next to leading orders of the (κ, χ)-expansion. The results of our calculations are $\eta=0.11$ (0.3 ± 0.3 [12]), $\kappa=0.4$, $\chi=0.28$, $\nu=s=0.66$.

The results obtained allow us to hope that the method of self-consistent equations can be effective for the study of the critical behaviour at the Anderson transition from the insulating to the conducting phase. This method yields quite reasonable values of the exponents from the point of view of the constraint $\nu\geqslant 2/d$ [4] already in the leading approximation of the (κ, χ)-expansion. We intend to calculate the value of ν in next-to-leading approximation of the (κ, χ)-expansion and hope that the inequality mentioned above will be satisfied.

We considered the regime without mixing (mixed propagator $D_{\psi\phi}=0$). If $D_{\psi\phi}\neq 0$, then there is no conformal invariance in the theory. In this case the conformal bootstrap method is useless for solving self-consistent equations, but one can calculate the critical exponents in the fashion described above for this regime, too. Preliminary calculations show that the $\psi\phi$-mixing is not possible at $K=0$ replicas.

We are grateful to Professor F Wegner and Professor A N Vasil'ev for fruitful discussions. One of the authors (YuMP) would like to thank Dr J R Honkonen for a critical reading of the manuscript and the Research Institute for Theoretical Physics, University of Helsinki, for hospitality.

References

[1] Schäfer L and Wegner F 1980 *Z. Phys.* B **38** 113
[2] Bernreuther W and Wegner F 1986 *Phys. Rev. Lett.* **57** 1383
[3] Wegner F 1989 *Nucl. Phys.* B **316** 663
[4] Chayes J T, Chayes L, Fisher D S and Spencer T 1986 *Phys. Rev. Lett.* **57** 2999
[5] Kravtsov V E, Lerner I V and Yudson V I 1988 *Zh. Eksp. Teor. Fiz.* **94** 255
[6] Polyakov A M 1969 *Zh. Eksp. Teor. Fiz.* **28** 553
Migdal A A 1969 *Zh. Eksp. Teor. Fiz.* **28** 1036
Parisi G and Peliti L 1971 *Lett. Nuovo Cimento* **2** 627
Parisi G 1972 *Lett. Nuovo Cimento* **4** 777
Bray A 1974 *Phys. Rev. Lett.* **32** 1413
[7] Kondor I, Temesvári T and Herényi L 1980 *Phys. Rev.* B **22** 1451
[8] Vasil'ev A N, Pis'mak Yu M and Honkonen J R 1981 *Teor. Mat. Fiz.* **46** 157
[9] Vasil'ev A N, Pis'mak Yu M and Honkonen J R 1981 *Teor. Mat. Fiz.* **47** 291
[10] Kramer B, Broderix K, MacKinnon A and Schreiber M 1990 *Physica* **167A** 163
[11] Fisher M E 1967 *Rep. Prog. Phys.* **30** 615
Zinn-Justin J 1979 *J. Physique* **40** 969
Ferer M, Moore M A and Wortis M 1971 *Phys. Rev.* B **4** 3954
[12] Soukoulis C M and Economou E N 1982 *Phys. Rev. Lett.* **52** 565